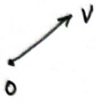
§4 Matrix Representation of Linear Maps. Change of Basis

This section reduces 2040 to 1030 by a process called "choosing a basis"

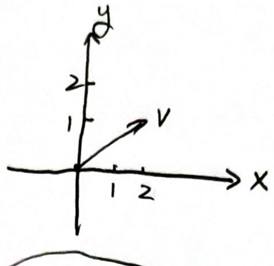
Coordinate free

(i) Vector spaces



$v \in V$

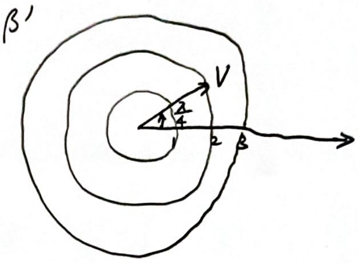
Pick a coordinate β (Cartesian)



$[v]_\beta = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{matrix} \leftarrow x \\ \leftarrow y \end{matrix}$

change of coordinate (basis).

Pick another coordinate β' (polar)



$[v]_{\beta'} = \begin{pmatrix} 2 \\ \frac{\pi}{4} \end{pmatrix} \begin{matrix} \leftarrow r \\ \leftarrow \theta \end{matrix}$

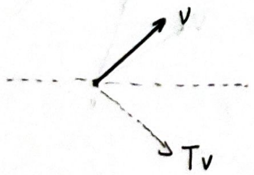
independent of coordinate ↑ essential information of an object 2040

one representation of an object. 1030

(ii) Linear maps

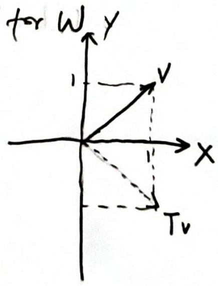
$V \xrightarrow{T} W = V$

v



Reflection by dotted line.

Pick a basis α for V, β for W



Choose $\alpha = \beta = \{\overset{(1,0)}{\overset{\shortparallel}{x}}, \overset{(0,1)}{\overset{\shortparallel}{y}}\}$.

$[v]_\alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $[Tv]_\beta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

T maps $\begin{pmatrix} 1 \\ 1 \end{pmatrix}_\alpha$ to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}_\beta$

so $[T]_\alpha^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

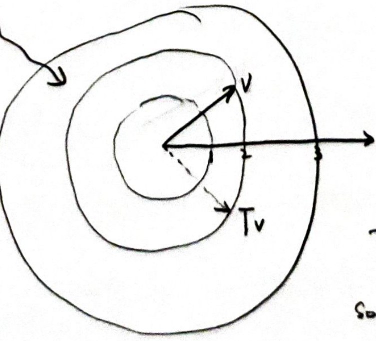
$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$[T]_\alpha^\beta \, [v]_\alpha = [Tv]_\beta$.

change of coordinate (basis)

Pick another basis α' for V and β' for W

Independent of coordinate ↑ essential information of a linear map 2040



Choose polar coordinate $\alpha' = \beta' = \{\overset{(1,0)}{\overset{\shortparallel}{r}}, \overset{(0,1)}{\overset{\shortparallel}{\theta}}\}$. $\theta \in \mathbb{R}/2\pi\mathbb{Z}$

$[v]_{\alpha'} = (\sqrt{2}, \frac{\pi}{4})$

$[Tv]_{\beta'} = (\sqrt{2}, -\frac{\pi}{4})$

↑ $-\frac{\pi}{4} = \frac{7\pi}{4}$

T maps $\begin{pmatrix} \sqrt{2} \\ \frac{\pi}{4} \end{pmatrix}$ to $\begin{pmatrix} \sqrt{2} \\ -\frac{\pi}{4} \end{pmatrix}$.

so $[T]_{\alpha'}^{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ -\frac{\pi}{4} \end{pmatrix}$.

$[T]_{\alpha'}^{\beta'} [v]_{\alpha'} = [Tv]_{\beta'}$

one representation of an object. 1030

Throughout this tutorial, we denote

V, W are finite dimension vector space over $\mathbb{F}$.

↑ we don't have $\infty \times \infty$ matrices.

$T: V \to W$ is a linear map.

⚠ By a basis we mean an ordered basis, i.e., the order matters. (eg. $\{e_1, e_2\} \neq \{e_2, e_1\}$ as basis).

Def. 4.1 (coordinate vector).

Choose a basis $\beta = \{e_1, \cdots, e_n\}$ for V ($n = \dim V$). Then for any $v \in V$.

$$v = \sum_{i=1}^{n} v_i e_i = (e_1\ e_2 \cdots e_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Write $[v]_\beta = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ called coordinate vector of v with respect to (w.r.t) basis β.

Remark. (i) This is well-defined as the linear combination exist and unique (because of basis).

(ii). This defines a natural map $[\ ]_\beta : V \longrightarrow \mathbb{F}^n$

$$v \longmapsto [v]_\beta = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Every choice of basis gives rise to a map $V \to \mathbb{F}^n$.

Def. 4.2. (matrix representation).

For $T: V \to W$. Choose a basis $\alpha = \{e_1, \cdots, e_n\}$ for V and ($n = \dim V$)

a basis $\beta = \{e_1', \cdots, e_m'\}$ for W. ($m = \dim W$) (n may not be equal to m).

Then we can represent T by a matrix $[T]_\alpha^\beta = A$ defined by.

$$A_{ij} = ([Te_j]_\beta)_i, \quad \text{or} \quad A = \begin{pmatrix} | & | & & | \\ [Te_1]_\beta & [Te_2]_\beta & \cdots & [Te_n] \\ | & | & & | \end{pmatrix} \quad \text{or}$$

$$Te_j = A_{1j} e_1' + A_{2j} e_2' + \cdots + A_{mj} e_m' = (e_1'\ e_2' \cdots e_m')\begin{pmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{pmatrix}$$

↖ j^{th} column.

Eg. 4.3 $T: \mathbb{F}^2 \to \mathbb{F}^3$, $(x, y) \longmapsto (x+3y,\ 2x+5y,\ 7x+9y)$.

(i) If we choose standard basis $\alpha = \{(1,0), (0,1)\}$ for $\mathbb{F}^2$, $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$ for $\mathbb{F}^3$.

then $T(1,0) = 1\cdot(1,0,0) + 2(0,1,0) + 7(0,0,1) \in \mathbb{F}^3$, $[T(1,0)]_\beta = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$

$T(0,1) = 3\cdot(1,0,0) + 5(0,1,0) + 9(0,0,1) \in \mathbb{F}^3$, $[T(0,1)]_\beta = \begin{pmatrix} 3 \\ 5 \\ 9 \end{pmatrix}$

so $[T]_\alpha^\beta = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$.

(ii). If we choose a basis $\alpha' = \{(1,2),(2,-1)\}$ of $\mathbb{F}^2$ and $\beta' = \beta = \{(1,0,0),(0,1,0),(0,0,1)\}$ for $\mathbb{F}^3$

Then

$$T(1,2) = 7(1,0,0) + 12(0,1,0) + 25(0,0,1) \rightsquigarrow [T(1,2)]_{\beta'} = \begin{pmatrix} 7 \\ 12 \\ 25 \end{pmatrix}$$

$$T(2,-1) = (-1)(1,0,0) + (-1)(0,1,0) + 5(0,0,1) \rightsquigarrow [T(2,-1)]_{\beta'} = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$$

so $[T]_{\alpha'}^{\beta'} = \begin{pmatrix} 7 & -1 \\ 12 & -1 \\ 25 & 5 \end{pmatrix}$

(iii) What if switch order?

If we choose $\alpha'' = \alpha' = \{(1,2),(2,-1)\}$ for $\mathbb{F}^2$, and $\beta'' = \{(0,0,1),(0,1,0),(1,0,0)\}$ for $\mathbb{F}^3$.

Then

$$T(1,2) = 25(0,0,1) + 12(0,1,0) + 7(1,0,0) \rightsquigarrow [T(1,2)]_{\beta''} = \begin{pmatrix} 25 \\ 12 \\ 7 \end{pmatrix}$$

$$T(2,-1) = 5(0,0,1) + (-1)(0,1,0) + (-1)(1,0,0) \rightsquigarrow [T(2,-1)]_{\beta''} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}$$

so $[T]_{\alpha''}^{\beta''} = \begin{pmatrix} 25 & 5 \\ 12 & -1 \\ 7 & -1 \end{pmatrix}$. Note that $[T]_{\alpha''}^{\beta''} \neq [T]_{\alpha'}^{\beta'}$

Ex: (i) $[T_1 \circ T_2]_\alpha^\beta = [T_1]_\gamma^\beta [T_2]_\alpha^\gamma$ (ii) $[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}$ if T is invertible. (iii) $[T]_\alpha^\beta [v]_\alpha = [Tv]_\beta$. written vertically

Change of basis

Q: Is there a systematic way to choose a basis so that $[T]_\alpha^\beta$ is $\left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right)$?

In particular, if $\dim V = \dim W$, is there a way to make $[T]_\alpha^\beta$ diagonal?

Ans: Consider the commutative diagram

$$\begin{tikzcd}
V \arrow[r, "I_V", "\cong"'] \arrow[d, "{[\ ]_{\alpha_0}}"] & V \arrow[r, "T"] \arrow[d, "{[\ ]_\alpha}"] & W \arrow[r, "I_W", "\cong"'] \arrow[d, "{[\ ]_\beta}"] & W \arrow[d, "{[\ ]_{\beta_0}}"] \\
\mathbb{F}^n \arrow[r, "{[I_V]_{\alpha_0}^{\alpha}}"'] & \mathbb{F}^n \arrow[r, "{[T]_\alpha^\beta}"'] & \mathbb{F}^m \arrow[r, "{[I_W]_\beta^{\beta_0}}"'] & \mathbb{F}^m
\end{tikzcd}$$

Eg 4.3. (Cont'd). $T(x,y)=(x+3y,\ 2x+5y,\ 7x+9y)$. $\alpha=\{(1,0),(0,1)\}$. $\beta=\{(1,0,0),(0,1,0),(0,0,1)\}$.

$$\begin{tikzcd}
V \arrow[r, "id", "="'] \arrow[rrr, bend left, "\text{canonical embedding in first 2 coordinates}"] \arrow[d, "{[\ ]_{\alpha_0}}"] & V \arrow[r, "T"] \arrow[d, "{[\ ]_\alpha}"] & W \arrow[r, "id", "="'] \arrow[d, "{[\ ]_\beta}"] & W \arrow[d, "{[\ ]_{\beta_0}}"] \\
\mathbb{F}^n \arrow[r, "{[Id]^\alpha_{\alpha_0}}", "{\begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}}"'] \arrow[rrr, bend right, "{[T]^{\beta_0}_{\alpha_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}"'] & \mathbb{F}^n \arrow[r, "{[T]^\beta_\alpha}", "{\begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}}"'] & \mathbb{F}^m \arrow[r, "{[Id]^{\beta_0}_\beta}", "{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 17 & -12 & 1 \end{pmatrix}}"'] & \mathbb{F}^m
\end{tikzcd}$$

column operations.

First: $\begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}^{-1}}_{=\begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -17 & 12 \end{pmatrix}$

row operations

Second: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 17 & -12 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -17 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

So $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 17 & -12 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}\begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

We try to find α_0 so that $[I_V]^\alpha_{\alpha_0} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$. Let $\alpha_0=\{e_1,e_2\}$.

$$[Id]^\alpha_{\alpha_0}\underbrace{\begin{pmatrix} | & | \\ [e_1]_{\alpha_0} & [e_2]_{\alpha_0} \\ | & | \end{pmatrix}}_{id\ !!!} = \begin{pmatrix} | & | \\ [Id(e_1)]_\alpha & [Id(e_2)]_\alpha \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ [e_1]_\alpha & [e_2]_\alpha \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ e_1 & e_2 \\ | & | \end{pmatrix}$$ as α is the standard basis.

Hence $e_1=(-5,2)$. $e_2=(3,-1)$.

We try to find β_0 so that $[Id]^{\beta_0}_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 17 & -12 & 1 \end{pmatrix} \Rightarrow [Id^{-1}]^\beta_{\beta_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -17 & 12 & 1 \end{pmatrix} = \left([Id]^{\beta_0}_\beta\right)^{-1}$. Let $\beta_0=\{e_1',e_2',e_3'\}$.

$$[Id^{-1}]^\beta_{\beta_0}\underbrace{\begin{pmatrix} | & | & | \\ [e_1']_{\beta_0} & [e_2']_{\beta_0} & [e_3']_{\beta_0} \\ | & | & | \end{pmatrix}}_{id\ !!!} = \begin{pmatrix} | & | & | \\ [Id(e_1')]_\beta & [Id(e_2')]_\beta & [Id(e_3')]_\beta \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ e_1' & e_2' & e_3' \\ | & | & | \end{pmatrix}$$

as β is standard basis and $Id^{-1}(e_i')=e_i'$

Hence $e_1'=(1,0,-17)$ $e_2'=(0,1,12)$. $e_3'=(0,0,1)$

You may check $[T]^{\beta_0}_{\alpha_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ under these basis, our calculation is right.

Remark. Actually in definition of T, (x,y) means $x\cdot(1,0)+y(0,1)$ so we can view it to be coordinates w.r.t. standard basis.

Q. Why do we define

$$[T]_\alpha^\beta = \begin{pmatrix} | & & | \\ [Te_1]_\beta & & [Te_n]_\beta \\ | & & | \end{pmatrix} \ ? \qquad \alpha=\{e_1,\cdots,e_n\}.\ \beta=\{e_1',\cdots,e_m'\}$$

Ans. We want

$$[Tv]_\beta = [T]_\alpha^\beta [v]_\alpha .$$

Indeed,

$$v = \sum_{i=1}^n v_i e_i = (e_1,\cdots,e_n)\overbrace{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}}^{[v]_\alpha}$$

$$\begin{aligned} Tv &= \sum_{i=1}^n v_i(Te_i) = (Te_1 \cdots Te_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\ &= (e_1' \cdots e_m')\underbrace{\begin{pmatrix} | & & | \\ [Te_1]_\beta & \cdots & [Te_n]_\beta \\ | & & | \end{pmatrix}\underset{[v]_\alpha}{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}}}_{\begin{pmatrix} | \\ [Tv]_\beta \\ | \end{pmatrix}} \end{aligned}$$

Hence we define $\begin{pmatrix} | & & | \\ [Te_1]_\beta & \cdots & [Te_n]_\beta \\ | & & | \end{pmatrix}$ to be $[T]_\alpha^\beta$.

Note that here we write basis element horizontally, and coordinate vertically.